The Chain Rule

Math 130 - Essentials of Calculus

15 March 2021

Differentiating a Composition

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Using Leibniz notation with $y = f(u)$ and $u = g(x)$, the chain rule becomes

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

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10. $k(x) = e^{\sqrt{t^2+2}}$
11. $y = \dfrac{r}{\sqrt{r^2 + 1}}$

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Using this, we can rewrite $b = \ln(e^b)$ so that

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Then, using the chain rule, we find that

$$\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x \ln b}] = e^{x \ln b} \ln b = b^x \ln b.$$

Example

Find the derivative of the following functions

1. $f(x) = 10^x$
2. $g(t) = 2^{3t}$

The Chain Rule with a Table

Example

Here is a table of values for f, g, f', and g'.

x	f(x)	g(x)	f'(x)	g'(x)
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

Find the following values

1. *If $h(x) = f(g(x))$, find $h'(1)$.*

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3. *If $F(x) = f(f(x))$, find $F'(2)$.*
4. *If $G(x) = g(g(x))$, find $G'(3)$.*